Basic properties of complex numbers

Argand Plane

![Argand Plane Diagram]

\[ |Z| = |OP| = \sqrt{x^2 + y^2} \]

Argument \( \Rightarrow \theta = \tan^{-1}\frac{y}{x} \)

Conjugate of Z

If \( Z = x + iy \)
then, conjugate of \( Z \), \( Z^{\cdot\cdot} \), is defined as,
\( Z^{\cdot\cdot} = x - iy \)
\( Z^{\cdot\cdot} \) is mirror image of \( Z \) on x-axis.

Properties of conjugate of Z

If \( Z = Z^{\cdot\cdot} \), then \( Z \) is purely real.
If \( Z = -Z^{\cdot\cdot} \), then \( Z \) is purely imaginary.

Properties of \( Z^{\cdot\cdot}_1 \) & \( Z^{\cdot\cdot}_2 \)

\[ Re(Z) = Re(Z^{\cdot\cdot}) \]
\[ Im(Z) = -Im(Z^{\cdot\cdot}) \]
\[ Z_1 + Z_2^{\cdot\cdot} = Z_1^{\cdot\cdot} + Z_2^{\cdot\cdot} \]
\[ Z_1 - Z_2^{\cdot\cdot} = Z_1^{\cdot\cdot} - Z_2^{\cdot\cdot} \]
\[ Z_1 Z_2^{\cdot\cdot} = Z_1^{\cdot\cdot} Z_2^{\cdot\cdot} \]
\[ (z_1^{\cdot\cdot} z_2^{\cdot\cdot}) = z_1^{\cdot\cdot} z_2^{\cdot\cdot} (Z_2 \neq 0) \]

Argument of Z

Argument of Z

\[ Arg(Z) = \tan^{-1}\frac{y}{x} \]
\[ -\pi \leq \theta \leq \pi \]
Arg(Z) = 0 or \( \pi \Rightarrow \) Z is pure Real
Arg(Z) = \( \pm \pi \) \( \Rightarrow \) Z is pure imaginary

**Properties of arg \((Z_1)\) & agr \((Z_2)\)**

\[ \text{arg}(Z_1Z_2) = \text{arg}(Z_1) + \text{arg}(Z_2) \]
\[ \text{arg}(Z_1Z_2) = \text{arg}(Z_1) - \text{arg}(Z_2) \]

**Agr (Z) and Agr \((Z^-\)***

\[ \text{arg}(-Z) = \text{Arg}(Z) \pm \pi \]
\[ \text{Arg}(iy) = \pi / 2 \text{ if } y > 0 \]
\[ -\pi / 2 \text{ if } y < 0 \]
\[ \text{Arg}(Z^-) = \pm \pi \]
\[ \text{Arg}(Z^-) = -\text{arg}(Z) = \text{arg}(iz) \]
\[ \text{Arg}(Z) = \pm \pi \Rightarrow \) Z is pure imaginary. \]

**Modulus of Z**

\[ |Z| \]
\[ |Z| = \sqrt{x^2 + y^2} \]

Properties of modules
if \(|Z| = 0\) the \( Z = 0 \)
\[ ZZ^- = |Z|^2 \]

**|Z_1| and |Z_2|**

\[ \text{Re}(Z) \leq |Z| \text{ & Img}(Z) \leq |Z| \]
\[ |Z_1Z_2| = |Z_1||Z_2| \]
\[ ||Z_1Z_2|| = |Z_1||Z_2| \]

**|Z_1+Z_2|**

\[ |Z_1+Z_2|^2 = (Z_1+Z_2)(Z_1+Z_2^-) \]
\[ = |Z_1|^2 + |Z_2|^2 + 2 \text{Re}(Z_1Z_2^-) \]
\[ |Z|_n = |Z|^n; n \in N \]
Euler's Theorem

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

\[ Z = r e^{i\theta} \]

where, \[ r = |Z| = x^2 + y^2 \]

Cos\(\theta\), Sin\(\theta\) as \(f(e^{i\theta})\)

\[ \cos \theta = e^{i\theta} + e^{-i\theta} \]

\[ \sin \theta = e^{i\theta} - e^{-i\theta} \]

De Moivre's Theorem

De Moivre's Theorem

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\]

Which holds true for any integer \(n\)

**FORMULA**

\[ \text{cis} \theta \]

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]

\[ \theta = \tan^{-1} \frac{y}{x} \]

\[ Z = r \left[ \cos \theta + i \sin \theta \right] \]

\[ Z = rcis \theta \]

**Cube Roots of Unity**

- \(1, w, w^2\)
\[ x^3 - 1 = 0 \]
has three roots
1, w, w^2
where, \( w = -1 + i3\sqrt{2} \)
\( w^2 = -1 - i3\sqrt{2} \)

- **Properties of cube roots of unity**

  (i) \( 1 + w + w^2 = 0 \)
  (ii) \( w^3 = 1 \)
  (iii) \( 1 + w^m + w^{2m} = 3 \)
      (if m is multiple of 3)
  (iv) \( 1 + w^m + w^{2m} = 0 \)
      (if m is not a multiple of 3)

- **Nth Roots of unity**

  - \( 1 \sqrt{n} \)
    
    \[ x_n = 1 \]
    
    Roots are
    
    \[ x = cos2K\pi n + i sin2K\pi n \]
    
    where \( K \) is an integer
    
    \( K = 0, 1, 2, \ldots \ldots n-1 \)

  - **Sum of nth roots of unity**

    \[ 1 + \alpha + \alpha^2 + \alpha^3 \ldots \alpha^{n-1} = 0 \]

  **Product of nth roots of unity**

  - \( 1 \cdot \alpha \cdot \alpha^2 \ldots \alpha^{n-1} = \{-1 \text{ if n is even}, \text{1 if n is odd}\} \)
  - If \( 1, \alpha, \alpha^2, \alpha^3, \ldots \ldots \alpha^{n-1} \) are \( n \)th roots of unity then
    
    \( (1 - \alpha)(1 - \alpha^2), \ldots (1 - \alpha^{n-1}) = n \)
The points represented by \( n \) \( n \text{th} \) roots of unity are located at the vertices of a regular polygon of \( n \) sides inscribed in a unit circle having centre at the origin, one vertex is being on the positive Real axis.

**Vectorial Representation of a complex number**

Vectorial representation of complex number

\[
\begin{align*}
\vec{oq} & = rq \ e^{i\theta + \phi} \\
\overrightarrow{PQ} & = \overrightarrow{OQ} - \overrightarrow{OP} \\
& = rq \ e^{i(\theta + \phi)} - rp \ e^{i\theta}
\end{align*}
\]
Triangle Properties

Centroid, In centre, or thocentre and circum centre of a triangle

Centroid "C" = \( Z_1 + Z_2 + Z_3 \)

Incentre "I" = \( aZ_1 + bZ_2 + cZ_3 \)

Orthocentre = \( z_1 \tan A + z_2 \tan B + z_3 \tan C \tan A + \tan B + \tan C \)

Rotation

Rotation of A vector

CONI'S Rule

if a complex No is Rotated by an angle \( \theta \) anticlockwise about its tail without changing its length then the new vector will be

\( Z_2 = z \left( \cos \theta + i \sin \theta \right) \)

\( \arg(Z_2) = \arg(Z) + \theta \)

Locus based on complex numbers

Straight line

\( a\bar{z} + az = b \)

\( a = \) complex number

\( b = \) real number

Circle

\( |z-a| = r \)

\( a = \) Complex number. (Centre)

\( r = \) Real number (Radius)

Arc of a circle

if \( z \) varies so that

\( \arg(z-az-a_1) = \phi \)

where \( \phi \) is a constant angle then \( z \) describes an are of a segment of circle on \( aa_1 \), containing an angle \( \phi \)

Ellipse

\( |Z-Z_1| + |Z-Z_2| = 2a \)

Hyperbola
\[|Z-Z_1|-|Z-Z_2|=2a\]

**Distance between** \(Z_1,Z_2\)

\[|Z_1-Z_2|\] represent the distance between \(Z_1\) and \(Z_2\)

**Excentres**

The excentres of the triangle (in the Argand plane), formed by \(z_1,z_2,z_3\) are given by,

(i) \(I_1=-az_1+bz_2+cz_3-a+b+c\)
(ii) \(I_2=az_1-bz_2+cz_3a-b+c\)
(iii) \(I_3=az_1+bz_2-cz_3a+b-c\), where
\[a=|z_2-z_3|, b=|z_3-z_1|, c=|z_1-z_2|\]

**Circumcentre**

The circumcentre of the triangle (in the Argand plane), formed by \(z_1,z_2,z_3\) is given by 
\[
\sum z_1 \overline{z_1} (z_2-z_3) + \sum |z_1| z (z_2-z_3) \overline{z_1} (z_1-z_2) 
\]

**Orthocentre**

The orthocentre of the triangle (in the Argand plane), formed by \(z_1,z_2,z_3\) is given by 
\[
\sum (z_1z_2 \overline{z_3}) + \sum |z_1| (z_2 \overline{z_3}) \overline{z_1} (z_2-z_3) 
\]

**Parallelogram**

If \(z_1,z_2,z_3,z_4\) are vertices of a parallelogram if \(z_1+z_3=z_2+z_4\)

**Square**

If \(z_1,z_2,z_3,z_4\) are the vertices of a square in that order, then

a) \(z_1+z_3=z_2+z_4\),
point of intersection of diagonals

b) \(|z_1-z_2|=|z_2-z_3|=|z_3-z_4|=|z_4-z_1|\)
all sides are equal
c) \(|z_1-z_3|=|z_2-z_4|\)
since, diagonals are equal
d) \((z_1-z_3)(z_2-z_4)\) is purely imaginary. Since, diagonals are perpendicular.

**The general equation of a line**

The general equation of a line in complex plane is \(a^*-z+a^*-b=0\), where \(b\) is a real number.
Equilateral triangle and Circumcenter

If the complex numbers $z_1, z_2, z_3$ be the vertices of an equilateral triangle and if $z_0$ be the circumcentre of the triangle, then $z_1 + z_2 + z_3 = 3z_0$

The general equation of a circle

General Equation of a Circle
$zz^* + az^* + a^*z + b = 0$ where $b$ is a real number.
The centre of the circle is $-a$ and its radius is $\sqrt{a^* - b}$

Condition for circle
$|z-z_1|^2 + |z-z_2|^2 = k (k \epsilon \mathbb{R})$
will represent a circle if $k \geq 12|z_1 - z_2|^2$

Concyclic Points
Four points $z_1, z_2, z_3$ and $z_4$ are concyclic if and only if
$(z_1 - z_3)(z_2 - z_4)(z_1 - z_4)(z_2 - z_3)$ is purely real.

Equality
Two complex numbers are equal if and only if both their real and imaginary parts are equal. In symbols:
$$z_1 = z_2 \iff (\text{Re}(z_1) = \text{Re}(z_2) \land \text{Im}(z_1) = \text{Im}(z_2)).$$

Ordering
Because complex numbers are naturally thought of as existing on a two-dimensional plane, there is no natural linear ordering on the set of complex numbers.[8]

There is no linear ordering on the complex numbers that is compatible with addition and multiplication. Formally, we say that the complex numbers cannot have the structure of an ordered field. This is because any square in an ordered field is at least 0, but $i^2 = -1$.

Multiplication and division in polar form

Formulas for multiplication, division and exponentiation are simpler in polar form than the corresponding formulas in Cartesian coordinates. Given two complex
numbers \( z_1 = r_1 (\cos \varphi_1 + i \sin \varphi_1) \) and \( z_2 = r_2 (\cos \varphi_2 + i \sin \varphi_2) \), because of the well-known trigonometric identities

\[
\begin{align*}
\cos(a) \cos(b) - \sin(a) \sin(b) &= \cos(a + b) \\
\cos(a) \sin(b) + \sin(a) \cos(b) &= \sin(a + b)
\end{align*}
\]

we may derive

\[ z_1 z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)) \].

In other words, the absolute values are multiplied and the arguments are added to yield the polar form of the product. For example, multiplying by \( i \) corresponds to a quarter-turn counterclockwise, which gives back \( i^2 = -1 \). The picture at the right illustrates the multiplication of

\[ (2 + i)(3 + i) = 5 + 5i. \]

Since the real and imaginary part of \( 5 + 5i \) are equal, the argument of that number is 45 degrees, or \( \pi/4 \) (in \textit{radian}). On the other hand, it is also the sum of the angles at the origin of the red and blue triangles are \( \arctan(1/3) \) and \( \arctan(1/2) \), respectively. Thus, the formula

\[ \frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3} \]

holds. As the \( \arctan \) function can be approximated highly efficiently, formulas like this—known as \textit{Machin-like formulas}—are used for high-precision approximations of \( \pi \).

Similarly, division is given by

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)) \].